MATH 223.3 2006-2007 (01) (03) (05)

QUIZ #2 - Solutions

Each question is worth 5 points - total = 25.
Integer scores *only*.

#1

The lines are parallel since a vector along each is $(3,4,1)$. Since $(1,0,-2)$ and $(-1,2,-5)$ are points on the lines, a second vector in the plane is $(1,0,-2)-(-1,2,-5)=(2,-2,3)$. A vector normal to the plane is

$$(3,4,1)\times(2,-2,3)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 4 & 1\\ 2 & -2 & 3\end{vmatrix}=(14,-7,-14),\quad \text{or,}\quad (2,-1,-2).$$

The equation of the plane is $0=(2,-1,-2)\cdot(x-1,y,z+2)=2x-y-2z-6$.

#2

Parametric equations for the line are $x=t$, $y=2t-5$, $z=10-3(t)-4(2t-5)=30-11t$. By solving each for t, symmetric equations are $x=\dfrac{y+5}{2}=\dfrac{z-30}{-11}$. A vector equation is $\mathbf{r}=(x,y,z)=(0,-5,30)+t(1,2,-11)$.

#3 **NOTE!** The “hat” notation “^” indicates that the vector has been normalized … i.e. scaled to a unit vector in the direction of the vector *under* the hat … thus $(\mathbf{v})^\wedge$ is a unit vector in the direction of $\mathbf{v}$… and $(\mathbf{v})^\wedge = \mathbf{v}/\,v$, where $v = \|\mathbf{v}\|$

Since x decreases along the curve, we set $x=-t$ for parametric equations, in which case $y=5+t$, $z=t^2-5-t$. A vector equation for the curve is $\mathbf{r}=-t\hat{\mathbf{i}}+(5+t)\hat{\mathbf{j}}+(t^2-t-5)\hat{\mathbf{k}}$, $-5\le t\le 0$. A tangent vector is $\mathbf{T}=\dfrac{d\mathbf{r}}{dt}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}+(2t-1)\hat{\mathbf{k}}$, and a unit tangent vector is

$$\hat{\mathbf{T}}=\frac{-\hat{\mathbf{i}}+\hat{\mathbf{j}}+(2t-1)\hat{\mathbf{k}}}{\sqrt{1+1+(2t-1)^2}}=\frac{-\hat{\mathbf{i}}+\hat{\mathbf{j}}+(2t-1)\hat{\mathbf{k}}}{\sqrt{4t^2-4t+3}}.$$

#4 **NOTE!** The "hat" notation "^" indicates that the vector has been normalized … i.e. made into a unit vector in the direction of the vector *under* the hat … thus $(\mathbf{T})^\wedge$ is a unit vector in the direction of **T,** and $(d\mathbf{T}^\wedge/dt)^\wedge$ is a unit vector in the direction of $d\mathbf{T}^\wedge/dt$ …etc

With parametric equations $x = -t$, $y = 5 + t$, $z = t^2 - t - 5$, (see Exercise 12.11–4),

$$\hat{\mathbf{T}} = \frac{(-1, 1, 2t - 1)}{\sqrt{1 + 1 + (2t - 1)^2}} = \frac{(-1, 1, 2t - 1)}{\sqrt{4t^2 - 4t + 3}}.$$

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(4t - 2)}{(4t^2 - 4t + 3)^{3/2}}(-1, 1, 2t - 1) + \frac{(0, 0, 2)}{\sqrt{4t^2 - 4t + 3}} \\ &= \frac{2}{(4t^2 - 4t + 3)^{3/2}}\left[-(2t - 1)(-1, 1, 2t - 1) + (4t^2 - 4t + 3)(0, 0, 1)\right] \\ &= \frac{2}{(4t^2 - 4t + 3)^{3/2}}(2t - 1, 1 - 2t, 2).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(2t - 1, 1 - 2t, 2)}{\sqrt{(2t - 1)^2 + (1 - 2t)^2 + 4}} = \frac{(2t - 1, 1 - 2t, 2)}{\sqrt{8t^2 - 8t + 6}}.$$

The direction of the binormal is

$$\begin{aligned}\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 2t - 1 \\ 2t - 1 & 1 - 2t & 2 \end{vmatrix} &= (2 + 4t^2 - 4t + 1)\hat{\mathbf{i}} + (4t^2 - 4t + 1 + 2)\hat{\mathbf{j}} + (-1 + 2t - 2t + 1)\hat{\mathbf{k}} \\ &= (3 - 4t + 4t^2)(\hat{\mathbf{i}} + \hat{\mathbf{j}}).\end{aligned}$$

Thus, $\hat{\mathbf{B}} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})/\sqrt{2}$.

#5 Same comments as for #3, #4.

From $\hat{\mathbf{T}} = \dfrac{(-4\sin t, 6\cos t, 2\cos t)}{\sqrt{16\sin^2 t + 36\cos^2 t + 4\cos^2 t}} = \dfrac{(-2\sin t, 3\cos t, \cos t)}{\sqrt{4 + 6\cos^2 t}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{6\cos t\,\sin t}{(4 + 6\cos^2 t)^{3/2}}(-2\sin t, 3\cos t, \cos t) + \frac{(-2\cos t, -3\sin t, -\sin t)}{\sqrt{4 + 6\cos^2 t}}.$$

At $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$, we may take $t = \pi/4$, in which case

$$\mathbf{N}(\pi/4) = \frac{3}{7\sqrt{7}}(-\sqrt{2}, 3/\sqrt{2}, 1/\sqrt{2}) + \frac{(-\sqrt{2}, -3/\sqrt{2}, -1/\sqrt{2})}{\sqrt{7}} = -\frac{4}{7\sqrt{14}}(5, 3, 1).$$

Hence, the principal normal at $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$ is $\hat{\mathbf{N}} = -\dfrac{(5, 3, 1)}{\sqrt{35}}$. Since a tangent vector at the point is $\mathbf{T}(\pi/4) = (-\sqrt{2}, 3/\sqrt{2}, 1/\sqrt{2}) = (-2, 3, 1)/\sqrt{2}$, the direction of the binormal at the point is

$$\mathbf{B}(\pi/4) = (-2, 3, 1) \times [-(5, 3, 1)] = -\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 3 & 1 \\ 5 & 3 & 1 \end{vmatrix} = -(0, 7, -21).$$

Thus, $\hat{\mathbf{B}}(\pi/4) = (0, -1, 3)/\sqrt{10}$.